Journal of Computational Physics **166,** 165–172 (2001)
doi:10.1006/jcph.2001.6645, available online at http://www.idealibrary.com on IDEAL®

# NOTE

## Chaos, Number Theory, and Computers

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Received January 19, 2000

A case study is presented in which the logistic map is shown to have anomalous, precision-dependent behavior. It is also demonstrated that using greater precision does not necessarily lead to greater accuracy and, in fact, the contrary can be true. Though the presention deals only with this particular map it illustrates that the actual math package being used should be examined and cannot be treated as a black box. 

*Key Words:* chaos theory; precision; computer simulation; logistic map.

### 1. INTRODUCTION

This study deals with the possibility of computer-induced artifacts in chaos studies. The analysis is structured around the quadratic iterator also known as the logistic map,

$$x_{n+1} = Ax_n(1 - x_n), \tag{1}$$

where $A \leq 4.0$ and $n = 0, 1, \ldots \infty$.

Nothing about the method of analysis is limited to this example but we know of no other examples in which similar results occur. Nonetheless the possibility of unwanted intrusions of computer arithmetic into numerical studies of any type does exist. Increasing precision does not necessarily increase accuracy, as is illustrated here.

The effect of computer round-off on the study of nonlinear systems is well documented [1, 2]. In general, the computer solution diverges exponentially from the "true" result. However, the shadowing theorem [3] states that the computer solution is near to, or shadows, a true solution; hence the computer solution itself is still useful [4]. Figure 1 (top) shows the divergence resulting from round-off of the iterates of the logistic equation. The "exact" solution is based on 28 digits of precision compared to 8 digits of precision for the

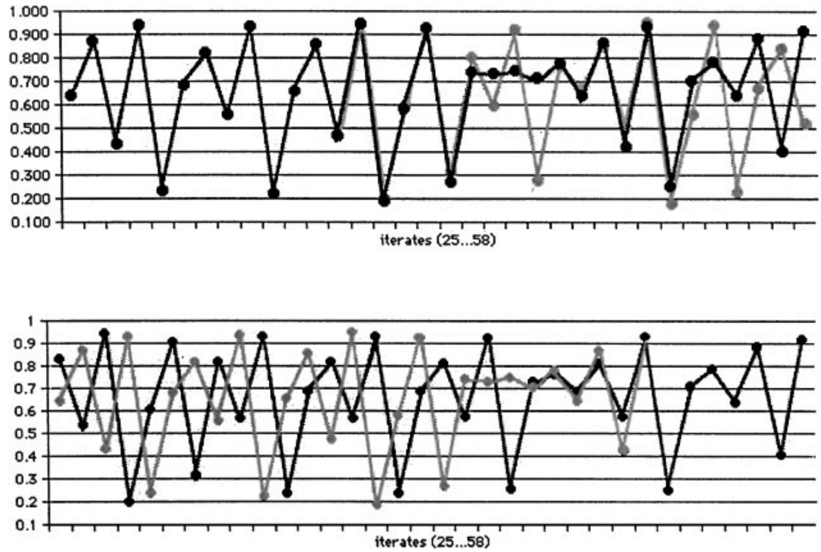


**FIG. 1.** Top: The true solution of the iterates of the logistic equation are shown in black and then grey. The extended black solutions show the effect of computer round-off. The initial seed in both cases is 0.2. Bottom: The "true" iterates of an initial seed of $x_0 = 0.217$ and $A = 3.8$, shown in gray, come in to shadow the rounded iterates of $A = 3.8$ and initial seed of 0.2 after around 50 iterations.

comparison solution. Both are based on $A = 3.8$ and an initial starting point of $x_0 = 0.2$. Figure 1 (bottom) shows how the iterates of another value of $x_0$ come in to shadow the precision 8 computer iterates of $A = 3.8$. Round-off, or finite precision, does not cause chaotic behavior [5] but can be used to predict the Lyapunov exponent. The Lyapunov exponent, for the one-dimensional quadratic iterator, $Ax(1-x)$, is given as $s(A) = -\lambda(A)/\ln(10)$, where $s(A)$ is the digit loss/gain per iteration as a function of the parameter, $A$, and $\lambda$ is the Lyapunov exponent. The growth in the error $\Delta E$ is proportional to $e^{\lambda}$, so a positive $\lambda$ produces an exponential decrease in precision and vice versa. If the parameter $A$ is in the chaotic regime, $\lambda$ will be positive and result in an increase in the error between iterations, moving continuously away from the current true trajectory. In this paper we present an important class of numbers for which this is not the case.

Most initial conditions, $x_0$, correspond to the same Lyapunov exponent, $\lambda$, which, along with shadowing, prevents finite digit arithmetic from masking the effects of chaos. This allows the loss of precision due to finite digit arithmetic to be viewed as an *ex-post-facto* shift in $x_0$, which does not alter the overall behavior of the iteration. For example, if the parameter $A$ is such that the iteration should produce a stable double population, the iteration will move toward the population with a decreasing error in proportion to $e^{\lambda}$, where $\lambda$ will be negative . However, if $A$ is in the chaotic regime, the iteration will move continuously away from the current true trajectory with an error proportional to $e^{\lambda}$, where $\lambda$ is positive. This last statement is not true for an important class of numbers, namely, numbers where positive $\lambda$ does not cause increasing error.

The present discussion will be limited to rational numbers, $p/q$, $p$ and $q$ both integer. Computers are capable of representing only rational numbers exactly. While the following holds strictly true for BCD (binary coded decimal) arithmetic, where the individual decimal digits are separately converted to their binary equivalents, typical binary representations produce results consistent with those presented here. The derivation presented here assumes that if the decimal representation of the entered number is longer than the specified precision of the computer, the number is truncated. For example, $1/7 = 0.142857142857...$ would be represented by 0.14285714 in precision 8. On the other hand it can be difficult to tell how the arithmetic is actually done. It may well be that although the specified precision is 8, calculations are actually performed using 10 digits and rounded to 8. For this reason, and for the convenience of the reader, a BCD emulation via JavaScript is directly accessable at `http://www.image-ination.com/logis.html`. Since JavaScript is dependent on browser evolution and new browsers may not always properly respect earlier versions of JavaScript, we also have available the same emulation written in Scheme. This has been tested on Unix, Windows, and Macintosh systems. The source is available at `http://www.image-ination.com/logistic.txt` and instructions at `http://www.image-ination.com/ReadMe.txt`. A runtime version for the Macintosh is available at `ftp://image-ination.com/Logistic.hqx`.

We might encounter an error in $x_0$ in one of two ways: either $x_0$ is inaccurate because of insufficient knowledge on our part or it cannot be represented without error by a computer. For example, if we know $x_0$ to be exactly $1/11$ and wish to use this initial value in a computer program we introduce an error, $x_0 - \overline{x}_0$, where $\overline{x}_0$ is $x_0$ as represented by the computer. The error in this case is

$$\Delta E_0 = 10^{-N}\,\mathrm{frac}(10^N(1/11)), \tag{2}$$

where $N$ is the precision of the computer and frac returns the fractional part of a number. One may attempt to avoid precision-induced error by using only rational arithmetic. This preserves the “true” trajectory but almost always exceeds the capacity of the computer in short order. Here we consider only one-dimensional system errors which propagate because of precision-induced errors in $x_0$.

## 2. RESULTS

In order for the quadratic iterator parameter, $A$, to be exactly represented by the computer it is taken to be of the form $q/p$, where $p$ contains only factors of 2 or 5. The fixed point of the iterator,

$$x_0 = (1 - 1/A), \tag{3}$$

becomes unstable at $A = 3$ and is replaced by a repeating double population which with increasing $A$ rapidly splits into populations of 4, 8, 16, etc., until chaos is reached. However, as expected, if rational arithmetic is used for $A = 3.5$ (i.e., $7/2$), where conventional decimal iteration would be expected to produce a repeating series of four populations, the fixed point $5/7$ calculated from Eq. (3) is metastable. This result follows from the fact that no information is lost between iterations when rational arithmetic is used and from the fact that fractions of the form $n/7$ ($n = 1, \ldots, 6$) have the property that either $n/7$ or $1 - (n/7)$ is even. This cancels the 2 in the denominator of $7/2$, causing the resulting iteration to always produce a member of the $n/7$ family; consequently, the information content does not grow unmanageably. This result would not be expected from finite precision decimal arithmetic but, surprisingly, using decimal arithmetic with a precision of 9 ($5/7 = 0.714285714$) or

7 ($5/7 = 0.7142857$) still produces the same metastable population after any number of iterations. However, for a precision of 8 ($5/7 = 0.71428571$), the single population is not stable. The Lyapunov exponent for precisions 9 and 7 is 0.4055 (and so they should not be, but are, metastable), but for precision 8 it is negative ($-0.81$) as expected, both as calculated in [1]. Clearly, the precision of the computer plays an important role in this instance as in many others.

Consider $A = q/p$ and $x_0 = 1 - p/q$. $\overline{x}_0$ will be $x_0$ as represented by the computer,

$$\overline{x}_0 = 1 - p/q - \epsilon_0(N, q, p), \tag{4}$$

where

$$\epsilon_0(N, q, p) \equiv 10^{-N} \operatorname{frac}(10^N(1 - p/q)). \tag{5}$$

Iterating $\overline{x}_0$ will produce a growth in the error such that

$$x_1 = q/p(1 - p/q - \epsilon_0(N, q, p))(p/q + \epsilon_0(N, q, p)) \tag{6}$$

$$= 1 - p/q + \epsilon_1(N, q, p), \tag{7}$$

where

$$\epsilon_1(N, q, p) = (q/p - 2)\epsilon_0(N, q, p) \tag{8}$$

and terms quadratic in $\epsilon_0$ have been ignored. Values of $A$ greater than 3 ensure that $\epsilon_1 > \epsilon_0$. This divergence from the stable solution will continue as the iteration moves away from the initial seed, $x_0 = 1 - p/q$. However, the finite precision of the computer calculation introduces an additional error, $\epsilon_2$, in the value of $x_1$,

$$\epsilon_2(N, q, p) = 10^{-N} \operatorname{frac}(10^N x_1). \tag{9}$$

If

$$\epsilon_2(N, q, p) = (q/p - 1)\epsilon_0(N, q, p), \tag{10}$$

then the propagation error will be canceled at each iteration by the corresponding truncation error. To see this, write $x_1$,

$$x_1 = \overline{x}_0 + \epsilon_0(N, q, p) + \epsilon_1(N, q, p) \tag{11}$$

$$= \overline{x}_0 + (q/p - 1)\epsilon_0(N, q, p). \tag{12}$$

For $\overline{x}_1$ to equal $\overline{x}_0$ requires that

$$(q/p - 1)\epsilon_0(N, q, p) = \epsilon_2(N, q, p) \tag{13}$$

or

$$\begin{aligned}(q/p - 1) \operatorname{frac}(10^N(1 - p/q)) \\ = \operatorname{frac}(10^N(\overline{x}_0 + (q/p - 1)10^{-N} \operatorname{frac}(10^N(1 - p/q))).\end{aligned} \tag{14}$$

Since $\bar{x}_0 = 1 - p/q$ and $\mathrm{frac}(m + x) = \mathrm{frac}(x)$ for $m$ integer, the above reduces to

$$(q/p - 1)\,\mathrm{frac}(10^N(1 - p/q)) = \mathrm{frac}((q/p - 1)\,\mathrm{frac}(10^N(1 - p/q))), \tag{15}$$

from which it is concluded that for $\bar{x}_1 = \bar{x}_0$, $(q/p - 1)\,\mathrm{frac}(10^N(1 - p/q))$ must be less than 1. This is a sufficient condition for the truncation error to cancel the propagation error.

For example, consider $p = 2$, $q = 7$ so that $x_0 = 5/7$. Since 10 is a primitive root mod 7,

$$10^6 \equiv 1 (\mathrm{mod}\ 7), \tag{16}$$

it follows [6] that the repeating fraction for $1/7$ has period 6 and that $m/7$, $m = 2, \ldots, 6$, is just a cyclically shifted version of the decimal representation of $1/7$. Therefore,

$$\mathrm{frac}(10^N 5/7) = m(N)/7, \tag{17}$$

where $m(N) = 1, \ldots, 6$ depending on the precision, $N$. For $(q/p - 1)\,\mathrm{frac}(10^N(1 - p/q)) < 1$ to be true requires

$$(q/p - 1)m(N)/7 < 1 \tag{18}$$

or

$$m(N) < 14/5, \tag{19}$$

which implies that $m(N)$ equals 1 or 2. For precision $N = 9$, $m(9) = 2$, and for precision $N = 7$, $m(7) = 1$, and therefore the condition is satisfied in both cases, but for $N = 8$, $m(8) = 4$, which does not satisfy the condition above. This explains the earlier observation about the stability of $5/7$ for $A = 7/2$ with precisions 9 and 7 but not 8. Any precision that produces an $m(N)$ of 1 or 2 will be metastable. The value of $m(N)$ is found from the periodic nature of the repeating fraction $1/7$ as the value for which $m(N)/7$ equals the fractional part of $5/7$ shifted $N$ digits to the left.

Consider now the general case, which requires that

$$(q/p - 1)\,\mathrm{frac}(10^N(1 - p/q)) < 1 \tag{20}$$

or

$$\mathrm{frac}(10^N(1 - p/q)) < p/(q - p) = 1/(A - 1). \tag{21}$$

Since only values of $A$ from 3 to 4 are of interest, a sufficient condition for the cancellation of errors to occur is

$$\mathrm{frac}(10^N(1 - p/q)) < 1/3. \tag{22}$$

If $q$ is chosen so that

$$10^{q-1} \equiv 1 (\mathrm{mod}\ q), \tag{23}$$

then the cancellation condition above becomes

$$m(N)/q < 1/3, \tag{24}$$

where $m(N) = 1, \ldots, q-1$. For any $q > 3$ this is automatically satisfied for at least some values of $N$.

For any $q/p$, $x_0 = 1 - p/q$, for which $10^{q-1} \equiv 1(\bmod q)$, the above guarantees the existence of a precision for which the metastable single population $x_0$ is maintained upon iteration. The first few $q$ values for which this is true [6] are 7, 17, 19, 23, 29, 47, 59, 61, 67. For example, $q = 19$ and $p = 5$ correspond to an $A$ value firmly within the chaotic regime yet with a metastable single population of 14/19 for at least some precisions. If $q$ is prime and $k$ in $10^k \equiv 1(\bmod q)$ is not equal to $q-1$, then the family of fractions defined by $m/q$ will be broken into $(q-1)/k$ sets with $k$ elements each [6]. For example, if $q = 11$ then $k = 2$ and there are then 5 sets each containing 2 of the possible 10 fractions. This grouping does not guarantee that the proper $m(N)$ is available to satisfy the condition

$$m(N)/q < 1/A - 1, \tag{25}$$

since $m(N)/q$ would correspond to only that set of fractions that contains the starting point $(q-p)/q$. However, for all primes less than 100 it is always true that the condition can be satisfied for some precision for any $p$ such that $3 < q/p < 4$ and $p$ contains only factors of 2 or 5.

Are other conditions that produce stability available? In general we can write $x_0 = (q-p)/q$. If $q$ contains only factors of 2 or 5 then $x_0$ is also a terminating fraction. Any precision, $N$, for which $N$ is greater than the decimal fraction length of $x_0$ and $A$ will produce a metastable single population. A method for choosing values of $q$ and $p$, where both $x_0$ and $A$ are terminating decimals, follows.

The desired values of $A$ are between 3 and 4, which means that $A$ is of the form $3.xyz\ldots$ ($M$ decimal digits). The initial value, $x_0$, is a number between 0 and 1 of the form $0.uvw$ ($N$ decimal digits). Choosing $x_0 = 1/A$ makes $x_1 = 1 - 1/A$, which, for the quadratic iterator, is a candidate for a metastable population; therefore $1/A$ is a suitable choice for an initial position. Let $B = 1/A$ and write them as $A' = A10^M$ and $B' = B10^N$, from which it follows that $A'B' = 10^Z$, where $Z = M + N$. Without loss of generality we can take $A' = 2Z$ or $A' = 5Z$, which implies that $B' = 5Z$ or $2Z$, respectively. Therefore, from an $A'$ the appropriate value of $A$ is determined by placing the decimal after the first digit of $A'$ while the associated starting value of $x$ is found from $B'$ by placing the decimal at the front of the string of digits. So, for $Z = 5$, we have $2^5 = 32$ and $5^5 = 3125$ with $A = 3.2$ and $x_0 = 1 - B = 1 - 0.3125 = 0.6875$ when $M = 1$, $N = 4$. Also, for $M = 3$, $N = 2$, and $Z = 5$ we get $A = 3.125$ and $x_0 = 1 - 0.32 = 0.68$.

To find appropriate values for $A$ (and hence $q$ and $p$) it is necessary that $2^Z$ or $5^Z$ or both have a leading digit of 3. The decimal length of $2^Z$ is given by $\lceil Z\log(2)\rceil$. Calculating

$$A = 2^Z/10^{Z\log(2)} - \mathrm{frac}(Z\log(2)) \tag{26}$$

returns an expression for $A$ from 0 to 10. Requiring $3 \le A < 4$ reduces this to

$$3 \le 10^{\mathrm{frac}(Z\log(2))} < 4, \tag{27}$$

or

$$\log(3) \leq \mathrm{frac}(Z \log(2)) < \log(4). \tag{28}$$

Writing $Z = 10n + 5$ ($n = 1, 2, \ldots$) makes it evident that all $Z$ ending in 5 up to and including $Z = 95$ satisfy the above inequality. For $5^Z$ the appropriate condition is

$$\log(3) \leq \mathrm{frac}(Z \log(5)) < \log(4), \tag{29}$$

and writing $Z = 10n + 8$ implies that any $Z$ ending in 8 up to and including $Z = 118$ satisfies the inequality.

## 3. CONCLUSION

While the effect of computer-induced error on modeling chaotic behavior of nonlinear systems is well understood, this paper has presented cases in which the expected computer influence is not observed. This variation from the expected behavior can be understood as arising from the precision the computer uses in its calculations and manifests itself in two contrasting ways: for certain terminating decimals the single solution may be seen as always metastable, given the proper arithmetic and precision, because no information is ever lost, or in the case of repeating decimals stability may be introduced where none should exist by canceling errors in the first order. A new study [7] shows that additional anomalous stabilities exist because of cancellations in the second order. These effects are shown to be direct consequences of the precision used in the calculation. Therefore, increased precision in the calculations does not necessarily translate into less computer influence upon the results obtained. There is no evidence that the results reported here translate to higher dimensional maps. Investigations of Arnold's cat map [8],

$$x_{n+1} = \mathrm{frac}(x_n + y_n) \tag{30}$$

$$y_{n+1} = \mathrm{frac}(x_n + 2y_n), \tag{31}$$

produced errors similar to those discussed above but no cancellations were discovered. Further, no general rules are known that will guide investigators in determining when to guard against the effects reported here. It seems clear, though, that investigators would be wise to check their calculations with various precisions and look for inconsistent results.

## ACKNOWLEDGMENTS

Much of the work reported here, and related work, was done utilizing PARI, a powerful research-oriented number-theoretic language, which among other things allows the utilization of a great variety of precisions. PARI was developed by C. Batut, D. Bernard, H. Cohen, and M. Oliver. Versions for a wide variety of operating systems are available from Megrez.ceremab.U-bordeaux.fr by anonymous FTP. The authors thank Victor Miller of the Institute for Defense Analyses Center for Communication Research, Princeton, NJ, for valuable suggestions, as well as James Wirth and James Joyce from East Carolina University Mathematics and Physics Departments, respectively.

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